A Logical Framework for Graded Predicates

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Abstract. In this position paper we present a logical framework for modelling reasoning with graded predicates. We distinguish several types of graded predicates and discuss their ubiquity in rational interaction and the logical challenges they pose. We present mathematical fuzzy logic as a set of logical tools that can be used to model reasoning with graded predicates, and discuss a philosophical account of vagueness that makes use of these tools. This approach is then generalized to other kinds of graded predicates. Finally, we propose a general research program towards a logic-based account of reasoning with graded predicates.

Keywords: Graded predicates · Vagueness · Mathematical fuzzy logic

1 Introduction

A contemporary view of reasoning goes beyond the analysis of argumentation in discourse and takes rationality as a broad phenomenon that encompasses competence not only at organizing discourse, but also at making decisions, and taking actions towards goals, in light of our beliefs and knowledge. A logic-based account of reasoning should recognize that all these aspects of rationality involve heavy use of graded properties. Indeed, predicates that are a matter of more-or-less (such as *red*, *old*, *tall*, or *rich*) are ubiquitous in most domains of discourse and everyday reasoning scenarios. They include vague predicates (such as the examples just mentioned), but also predicates with sharply-defined boundaries. Take, for example, the predicate *acute angle* whose extension is exactly the set of angles strictly smaller than a right angle, and yet its instances admit mutual comparison: if α and β are angles of, respectively, 30° and 89°, it is true that both are acute angles, and it also makes sense to assert that α is strictly more acute than β. Non-graded all-or-nothing properties are actually rare in everyday communication and usually belong to quite restricted domains of discourse (typically, mathematics and other sciences as well as legal discourse).

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Arguably, graded properties are epistemologically necessary, as they gather together many similar notions that would otherwise collapse the conceptual system (and the language) with too many properties (and predicates). It is necessary for reasons of economy to reason with one predicate *red*, instead of having infinitely many predicates, for each possible level in the colour spectrum (or as many as the human eye can distinguish). Reasoning with such graded properties is successfully and correctly carried out in many contexts (notwithstanding the fact that natural language has enough devices to provide higher levels of precision whenever necessary).

We view Logic as the science of correct reasoning and, as such, we expect it to provide us with the formal means to deal with all forms of valid consequence that can potentially be carried out by rational beings. During most of its history as a formal science, Logic has tried to explain correct reasoning by means of the classical paradigm based on the bivalence principle. Despite its many merits and achievements, this approach does violence to many properties, forcing sharp all-or-nothing definitions (splitting graded properties into many binary ones) in contexts that normally do not require them. Indeed, natural language allows satisfactory communication and correct reasoning using graded predicates. The classical logical analysis seems, therefore, too artificial—too detached from actual reasoning.

On the one hand, there have been several attempts in philosophical logic and analytic philosophy at understanding vague predicates and their potential for generating logical paradoxes—although most of these attempts do not treat vague predicates as graded. On the other hand, Mathematical Fuzzy Logic (MFL) was proposed [17] as a study of many-valued logical systems able to handle graded properties (and related notions of partial truth, vagueness, fuzziness, imprecision, etc.). It has attracted a considerable number of researchers who have mostly disregarded its original motivations and focused on developing a deep and extensive corpus of mathematical results (see e.g. [6]), covering all technical aspects of such logical systems. Philosophers of vagueness have often attacked MFL as an inadequate framework for dealing with vagueness, based on allegations that usually disregard most of the mathematical development of MFL and focus on a few characteristics of the logical systems that were proposed at the beginning of the field. However, there has been a recent philosophical account of vague predicates [33] that treats them as graded, employs the modern logical machinery of MFL, and offers good answers to the traditional arguments against degree-based approaches.

In this paper we will defend a logical approach to reasoning with graded predicates that goes beyond that offered in [33] by considering other graded predicates besides vague ones. The structure of the paper is as follows. After this Introduction, Sect. 2 discusses the different kinds of graded predicates that we want to model. Section 3 presents a brief up-to-date account of mathematical fuzzy logic and Sect. 4 shows how it can be used to provide a satisfactory explanation of vague predicates and a solution to the paradoxes they generate. Section 5 discusses possible ways of modelling other kinds of graded predicates by means of the tools of MFL and Sect. 6 proposes a general program, extending the first steps outlined in this paper, to develop a full logic-based account of reasoning with graded predicates.

2 Graded Predicates

The essential feature of graded predicates is that they may apply with different intensities to different objects. If F is a graded unary predicate and a and b are objects in the domain of discourse relevant for F, it may happen that a is strictly more F than b, i.e. the degree of F-ness of a is greater than that of b; it could also be the other way around; or a and b could be equal; or they could be incomparable. All these possible comparisons do not entail the existence of any numerical scale, but only a purely ordinal notion of degree. Also, there are graded predicates of any higher arity, characterized in the same way, that apply to tuples instead of single individuals.

We may distinguish the following different kinds of graded predicates:[1]

1. **Classical predicates:** As an extreme case of our classification we must include the classical predicates. They obey the bivalence and excluded middle principles and hence yield a perfect division of the domain into the elements that satisfy the predicate and those that do not. They are a limit case of graded predicates that admit only two degrees. Classical predicates correspond to sharply-defined all-or-nothing properties and are ideal for analysing reasoning in domains that typically employ such notions, for example mathematics or legal discourse. However, they have often been abused to model other kinds of graded properties in an unnatural way.
2. **Vague predicates:** Vague predicates exhibit three surface characteristics: (a) their extension has blurry boundaries, (b) they have borderline cases (objects such that we can neither confidently assert nor confidently deny that they fall under the predicate), and (c) they generate *sorites* paradoxes, as follows. A sorites series for a predicate F is a series of objects $x_0, x_1, \ldots, x_n$ such that:
   - F definitely applies to x_0
   - F definitely does not apply to x_n
   - for each $i < n$, the objects x_i and x_{i+1} are extremely similar in all respects relevant to the application of F.

Such a series generates the following argument: (1) x_0 is F; (2) for each $i < n$, if x_i is F, then so is x_{i+1}; therefore x_n is F. When F is vague, this argument becomes a logical paradox, because it has the form of a valid argument whose first premise is clearly true and whose second premise also seems true, and yet its conclusion is clearly false.

Typical examples of vague predicates are those mentioned in the Introduction: *red*, *old*, *tall* and *rich*. Vague predicates can be subdivided into **linear**

[1] This classification is a modification of that presented by Paoli in [26,27].

(or **unidimensional)** and **nonlinear** vague predicates. The application of a linear vague predicate to an object depends only on the extent to which the object possesses some underlying attribute, which varies along a single dimension. For example, (once we fix a context) whether someone is 'tall' depends only on her height (and heights vary along a single dimension) and whether someone is 'old' depends only on his age (and ages vary along a single dimension). By contrast, the application of a nonlinear vague predicate to an object does not depend only on the position of that object along a single dimension. Some (perhaps all) nonlinear vague predicates are **multidimensional**: for example, whether an object is 'red' depends on its position in a three-dimensional colour space—that is, it depends on its position along three different dimensions (e.g. hue, saturation and brightness). Arguably, there is also a second kind of nonlinear vague predicate: one whose application conditions *cannot* be factored into a series of linear dimensions. For example, some might argue that 'beautiful' is such a predicate. We do not take a position either way on whether the nonlinear vague predicates are completely exhausted by the multidimensional vague predicates (i.e. on whether the class of nonlinear nonmultidimensional vague predicates is empty). Note that nonlinear vague predicates may have *incomparable* instances: for example, it may be possible to come up with two individuals such that there is no way to determine who is more clever, because they are clever in different ways.

Vagueness has been clearly distinguished from other phenomena (such as uncertainty, context sensitivity, ambiguity, and generality) and has been addressed by several competing theories (see e.g. [14,21,22,28,29,33,34,36]). In Sect. 4 we will summarize a degree-based treatment of vague predicates.

3. **Graded precise predicates:** These are predicates that have sharply defined limits but that, unlike classical predicates, admit more than two degrees of application. An example, already mentioned in the Introduction, is the unary predicate *acute angle*: it is sharply defined (as applying to angles strictly smaller than a right angle) but it applies with different intensities to different acute angles (an angle of 30° is more acute than an angle of 89°, although both are acute) and also to different non-acute angles (an angle of 170° is less acute than an angle of 91°, although both are non-acute). Other sciences also employ graded precise predicates: for example *acid* and *base* in chemistry, defined as having a pH smaller (resp. greater) than 7. Finally, to give an example from legal language, consider the predicate *guilty*. The judicial system does not want borderline cases, and will do everything it takes to prevent them and always declare an accused person either guilty or not. However, there are different degrees of guilt, which translate into more or less severe sentences.

Their well-defined limits save graded precise predicates from the difficulties of vagueness (in particular, the generation of sorites paradoxes), but they are still quite different from classical predicates and require a different logical treatment.

3 Mathematical Fuzzy Logic

Petr Hájek founded MFL [17] as an attempt to provide solid logical foundations for fuzzy set theory and its engineering applications. Among other motivations, fuzzy set theory had been explicitly proposed as a mathematical apparatus for dealing with vagueness and imprecision, but it lacked a focus on syntactical formalization of discourse and a notion of logical consequence, thus keeping it far from being a logical study of reasoning under vagueness. Hájek and his collaborators developed MFL as a genuine subdiscipline of Mathematical Logic, specializing in the study of certain many-valued logics.

The first examples of fuzzy logics were two many-valued propositional systems that had been studied already for quite some time before the inception of fuzzy sets: Łukasiewicz [24] and Gödel–Dummett logics [11]. Both were considered fuzzy logics because—similar to the definition of membership functions in fuzzy sets—they were semantically defined as infinitely-valued logics taking truth-values in the real unit interval $[0, 1]$. But they had more characteristics in common: a language with conjunction $\wedge$ and disjunction $\vee$ respectively interpreted as the operations minimum and maximum, constants for (total) falsity and truth $\overline{0}$ and $\overline{1}$ respectively interpreted as the values 0 and 1, an implication $\rightarrow$, and, in the case of Łukasiewicz, another conjunction connective (fusion) & satisfying the following residuation law with respect to the implication, for each $a, b, c \in [0, 1]$ (in the case of Gödel–Dummett logic it is satisfied by $\wedge$):

$$a \,\&\, b \leq c \text{ if, and only if, } a \leq b \rightarrow c.$$

Both these operations, used to interpret conjunctions, are particular instances of binary functions called *triangular norms* (or *t-norms* for short): binary commutative, associative, monotone functions on $[0, 1]$; and moreover they are both continuous, which guarantees the existence of a binary function satisfying the residuation law. Therefore, Hájek and other MFL researchers started proposing alternative $[0, 1]$-valued logics by keeping the interpretation of $\wedge$, $\vee$, $\overline{0}$ and $\overline{1}$ as in the previous systems, but taking other continuous t-norms for & and their corresponding residuum for $\rightarrow$ [5,17]. It was later observed that the necessary and sufficient condition for a t-norm to have a residuum was not continuity, but just left-continuity. This motivated the introduction, by Esteva and Godo, of MTL [12], a weaker logic that was later proved to be complete w.r.t. the semantics given by all left-continuous t-norms and their residua [19]. Therefore, MTL was proposed as a basic fuzzy logic upon which other fuzzy logics could be obtained as axiomatic extensions.

Besides their intended t-norm-based semantics over $[0, 1]$ (also called *standard* semantics), all these fuzzy logics were also given an algebraic semantics based on classes of MTL-algebras, that is, structures of the form $\boldsymbol{A} = \langle A, \wedge, \vee, \&, \rightarrow, \overline{0}, \overline{1}\rangle$ such that

- $\langle A, \wedge, \vee, \overline{0}, \overline{1}\rangle$ is a bounded lattice
- $\langle A, \&, \overline{1}\rangle$ is a commutative monoid

– for each $a, b, c \in A$ we have

$$a \mathbin{\&} b \leq c \quad \text{iff} \quad b \leq a \to c \qquad \text{(residuation)}$$
$$(a \to b) \vee (b \to a) = \overline{1} \qquad \text{(prelinearity)}$$

We say that an MTL-algebra is:

– *Linearly ordered* (or an MTL-chain) if its lattice order is total.
– *Standard* if its lattice reduct is the real unit interval $[0, 1]$ with its usual order.

Note that in a standard MTL-algebra & is interpreted by a left-continuous t-norm and → by its residuum—and vice versa: each left-continuous t-norm fully determines its corresponding standard MTL-algebra.

MTL is an algebraizable logic in the sense of [3] and the variety of MTL-algebras is its equivalent algebraic semantics. Thus each finitary extension of MTL (like all the other logics mentioned so far) also has an equivalent algebraic semantics which is a corresponding subquasivariety of MTL-algebras. Conversely, given any subquasivariety $\mathbb{K}$ of MTL-algebras, the corresponding finitary extension L of MTL is obtained by setting that for each set of formulas Γ and each formula φ, $\Gamma \vdash_{\mathrm{L}} \varphi$ iff for each algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \&, \to, \overline{0}, \overline{1}\rangle \in \mathbb{K}$ and each $\boldsymbol{A}$-evaluation e we have: if $e(\psi) = \overline{1}$ for each $\psi \in \Gamma$, then $e(\varphi) = \overline{1}$.

It soon became clear that fuzzy logics were closely related to substructural logics; indeed it was proven that MTL is the axiomatic extension of the logic $\mathrm{FL_{ew}}$ (the full Lambek logic with exchange and weakening, see e.g. [15]) obtained by adding the prelinearity axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$ [13]. Several papers have considered weaker fuzzy logics as extensions of other substructural logics:

(a) By dropping *commutativity* of conjunction Petr Hájek obtained a system, psMTL^r [18], which is an axiomatic extension of $\mathrm{FL_w}$ and was proven to be complete with respect to the semantics on non-commutative residuated t-norms [20].
(b) By removing *integrality* (i.e. not requiring the neutral element of conjunction to be maximum of the order) Metcalfe and Montagna proposed the logic UL which is an axiomatic extension of $\mathrm{FL_e}$ with bounds and was, in turn, proven to be complete with respect to left-continuous uninorms (that is, a generalization of t-norms that allows the neutral element to be any element $u \in [0, 1]$) [25].
(c) By removing *associativity* (i.e. not requiring conjunction to be interpreted by an associative operation) as well as commutativity and integrality, one obtains a very weak fuzzy logic SL^ℓ which extends the non-associative Lambek logic [16,23] and is still complete with respect to models over $[0, 1]$. The axiomatization and completeness theorems for this logic and for systems obtained with other combinations of the properties (associativity, integrality, and commutativity) are presented in [7,8].

All these fuzzy logics, weaker than MTL, are still algebraizable in the sense of [3] and their algebraic counterparts are classes of lattice-ordered residuated unital groupoids (not necessarily associative, commutative, or integral) in which the semantical consequence relation has to be defined in a more general way than before. More precisely, if $\mathbb{K}$ is a class of such algebras, Γ is a set of formulas and φ is a formula, $\Gamma \vdash_{\mathrm{L}} \varphi$ iff for each algebra $\boldsymbol{A} \in \mathbb{K}$ and each $\boldsymbol{A}$-evaluation e we have: if $e(\psi) \geq \overline{1}$ for each $\psi \in \Gamma$, then $e(\varphi) \geq \overline{1}$. Therefore, in these fuzzy logics the interpretation of the constant $\overline{1}$ is not the only relevant truth-value when it comes to defining consequence, but all the elements greater than $\overline{1}$; that is, the truth-preserving definition of consequence (usual in algebraic logic) uses in any algebra $\boldsymbol{A}$ the following set of *designated truth-degrees*: $D = \{a \in A \mid a \geq \overline{1}\}$.

A general property shared by all the mentioned algebraic semantics for fuzzy logics, from Łukasiewicz and Gödel–Dummett logics to these weaker systems, is that each algebra can be represented as a subdirect product of chains, i.e. can be embedded into a product of linearly ordered algebras in such a way that each projection is surjective. Therefore, all these logics are complete with respect to the semantics given just by linearly ordered algebras (and, for many prominent logics this gets even better, as we have mentioned, because they are complete w.r.t. standard chains). Based on this fact it has been argued that the only essential feature of fuzzy logics is that they are the logics of chains [2,8].

The field of research determined by this wide family of logics has attracted many researchers who have extensively carried out for MFL a typical agenda of mathematical logic: proof theory, model theory, modalities, first and higher order formalisms, axiomatic fuzzy set and fuzzy class theories, recursion and complexity, functional representation, different kinds of semantics, connections with other areas of Mathematics, applications to Philosophy, etc.; see e.g. the handbook series [6] and references therein.

4 A Degree-Based Account of Vagueness

The fundamental questions that a theory of vagueness should answer are: (1) What is the meaning (semantic value) of a vague predicate? and (2) How should we reason in the presence of vagueness? As part of answering these questions, a theory of vagueness should solve the sorites paradox—and this in turn involves two tasks: (i) Locate the error in the sorites argument: the premise that isn't true or the step of reasoning that is incorrect. This part of the solution should fall out of the answers to (1) and (2) above. (ii) Explain why the sorites argument for a vague predicate is a paradox rather than a simple fallacy: that is, provide an explanation of why competent speakers find the argument compelling but not convincing—why they initially go along with the reasoning but are still not inclined to accept the conclusion.

The simplest fuzzy answer to (1) is that the meaning of a vague predicate is a fuzzy set. However, this simple answer is inadequate. For it is generally accepted that language is a human artefact: the sounds we make mean what they do because of the kinds of situations in which we (and earlier speakers) have made (and would make) those sounds. This generates a constraint on any theory of vagueness: if the theory says that vague predicates have meanings of such-and-such a kind (e.g. fuzzy sets), then we must be able to satisfy ourselves that our past and present usage (and usage dispositions) could indeed determine such meanings for actual vague predicates. However it seems that usage and usage dispositions do not suffice to pick out a single fuzzy set—a particular function from objects to $[0, 1]$—as the extension of 'is tall' (and similarly for other vague predicates). For this reason, Smith [33] proposed *fuzzy plurivaluationism.* Instead of each vague discourse being associated with a unique intended fuzzy model, the plurivaluationist idea is that each vague discourse is associated with multiple acceptable fuzzy models. The acceptable models are all those that our usage and usage dispositions do *not* rule out as being incorrect interpretations of our language (e.g. an interpretation that does not map persons generally agreed to be paradigmatic instances of 'tall' to 1 is incorrect). On this view, a fuzzy set is the right *kind* of thing to be the meaning of a vague predicate—but there is not, in general, just one fuzzy set that is the uniquely correct meaning of a vague predicate in ordinary discourse. Rather, there are many fuzzy sets—one in each acceptable model—each of which is an equally correct meaning.

The answer that we propose to question (2) is that we should reason in the presence of vagueness in accordance with some system of MFL—although not necessarily the same system in every context: different reasoning scenarios may require different logics (see Sect. 6 below for more details).

Suppose we have a sorites series $x_0, \ldots, x_n$ for the predicate F and the associated sorites argument:

$$Fx_0, \quad Fx_0 \to Fx_1,\ Fx_1 \to Fx_2,\ \ldots,\ Fx_{n-1} \to Fx_n \qquad \therefore Fx_n.$$

If we employ Łukasiewicz logic with a definition of consequence as preservation of degree 1, then we get the following solution to the sorites. (i) The problem with the argument is that, although it is valid, it is unsound: it is not the case that every premise is true to degree 1. (ii) The argument is nevertheless compelling because all the premises are either true to degree 1 or *very nearly* true to degree 1, and in ordinary reasoning contexts we tend to apply a useful approximation heuristic that involves rounding very small differences up or down—hence we go along with the premisses, even though they are not all, strictly speaking, fully true.

Two key arguments in favour of this theory of vagueness are as follows. First, no other theory can solve the sorites in an equally satisfactory way: all other extant theories are forced to attribute ad hoc, implausible mistakes to ordinary reasoners to explain why they go along with the sorites reasoning [30]. Second, no other theory fits with our best understanding of what vagueness fundamentally consists in. In Sect. 2 we introduced vague predicates via three characteristics: blurry boundaries, borderline cases and sorites susceptibility. This can be compared to explaining what water is by saying it's a clear potable liquid that falls as rain and boils at 100 °C: this helps someone who doesn't know what water is to identify samples of it, but it still leaves open the question of the underlying nature or essence of water—of what water fundamentally *is*, that explains why it has these characteristics. The same goes for vagueness: it would be desirable to understand its fundamental nature and explain *why* it has the three surface characteristics. Smith [32,33] has argued that a predicate F is vague iff it satisfies the following Closeness principle:

> If x and y are very similar in respects relevant to the application of F, then Fx and Fy are very similar in respect of truth.

This yields explanations of why vague predicates have their surface characteristics: i.e. assuming only that a predicate P satisfies Closeness, we can derive that P must have blurry boundaries and borderline cases and generate sorites paradoxes. Furthermore, only theories of vagueness that admit degrees of truth can allow that there exist predicates that satisfy Closeness. This, then, is a strong reason for accepting fuzzy theories of vagueness, which do admit degrees of truth.

Two key arguments against fuzzy theories of vagueness are as follows. First, there is the artificial precision objection, that it is implausible to associate each vague predicate in natural language with a *particular* function that assigns a unique real number to each object. Fuzzy plurivaluationism avoids this objection, however, as it associates each vague predicate with many such functions (one per admissible model). Second, there is the truth-functionality objection, that fuzzy theories are incompatible with ordinary usage of compound propositions in the presence of borderline cases. However, this objection is based on an outdated understanding of fuzzy logics as having only very limited resources—for example, minimum and maximum as the only possible interpretations of conjunction and disjunction [31].

5 Modelling Graded Predicates in MFL

In the previous section we have seen that the algebraic semantics of many prominent fuzzy logics (such as Łukasiewicz, Gödel–Dummett, and other t-norm-based logics), though it has only one designated element on each algebra, is already powerful enough to provide a model of vagueness. This unique designated element (the maximum value of the lattice order, the number 1 on $[0,1]$-valued models) represents full truth and plays two important roles: it is the value used for the truth-preserving definition of logical consequence, and it is the neutral element of the operation that interprets the conjunction &.

We now sketch a proposal for modelling reasoning with graded predicates that exploits a greater part of the power of MFL. Indeed, we consider models for the weaker fuzzy logics mentioned in Sect. 3 where the neutral element of conjunction is an element $u \leq 1$, and the set of designated values that define the truth-preserving notion of consequence is $D = \{a \mid a \geq u\}$, not necessarily a singleton.

Vague Predicates: The usage of non-integral algebras of truth-values gives an interesting complement to the theory of vagueness explained in Sect. 4, that allows one to distinguish between different clear instances of a vague predicate. Take, for example, the predicate *tall* (and assume that we have already fixed a particular context of application). From a fuzzy plurivaluationist perspective, such a predicate admits many models which should all agree on clear instances and clear non-instances, and may assign degrees to borderline cases in different ways. Take individuals a, b, c, d, and e with respective heights of 2.1, 1.87, 1.78, 1.63, and 1.58 m. All models will agree that a and b are tall and that d and e are not tall, while c is a borderline case that will receive different degrees of tallness in different models. Algebraic models of MTL (and its extensions) have only one designated value for truth (1) and one for falsity (0), so the mentioned clear cases will only take these values. If T is a unary predicate symbol for *tall*, then the formulas Ta and Tb will be evaluated to 1, while Td and Te will be evaluated to 0, in symbols: $\|Ta\| = \|Tb\| = 1$ and $\|Td\| = \|Te\| = 0$. However, if we take instead an algebraic model of UL, defined for example by a left-continuous uninorm, then the set of designated elements is the interval $[u, 1]$, where u is the neutral element (the interpretation of $\overline{1}$). This provides a finer model for the vague predicate that allows one to make distinctions among clear cases, and clear non-cases, that is, both a and b are definitely tall, hence $\|Ta\|, \|Tb\| \in [u, 1]$, but a is much taller than b, which can be captured in the model by requiring $\|Ta\| > \|Tb\|$; hence the identification of all clear cases in one truth-value enforced by MTL and its extensions is no longer necessary. Similarly with the cases that are definitely not tall. Where f is the interpretation of $\overline{0}$, the set of degrees $[0, f]$ gives a whole range to interpret clear non-cases, in particular: $\|Td\|, \|Te\| \in [0, f]$ with $\|Te\| < \|Td\|$. This suggests, as already pointed out by Paoli [26], the following revision of the Closeness principle:

> x and y are very similar in F-relevant respects if, *and only if*, Fx and Fy are very similar in respect of truth.

Paoli uses this revised principle to argue that vague predicates can be better interpreted in models that have more than one truth-degree for clear cases, and more than one for clear non-cases. However, his proposal is restricted to the algebraic models of Casari's comparative logic [4]. We believe that such a restriction is not flexible enough, because for example it excludes non-commutative or non-associative interpretations of residuated conjunction, which are necessary in some reasoning scenarios.

On the other hand, algebraic models of fuzzy logics can always be decomposed into linearly ordered components (technically, by means of subdirect representation). This property allows us to account for the fact that many (maybe all) vague predicates depend on underlying parameters that vary on linear scales. Furthermore, if there are any vague predicates that cannot be explained from a set of parameters that vary on a linear scale of degrees—that is, if there are nonlinear nonmultidimensional vague predicates—they can still be modelled in a degree-based approach if we enhance somewhat the logical framework and allow for systems that do not enjoy completeness (and subdirect decomposition) w.r.t. chains. Algebraic logic offers methods to build algebraic models for any non-classical logic. In particular, if the logic has a reasonable implication connective, it induces an order relation in its algebraic models [9,10] and, hence, such algebras can be seen as (not necessarily linearly ordered) scales of degrees adequate for modelling such predicates.

Graded Precise Predicates: These predicates also admit models on algebras of fuzzy logics, but with an important restriction of the evaluation functions to account for the fact that there are no borderline cases. Take, for instance, the predicate *acute angle*, represented by the unary predicate symbol A. Consider again a model defined by a left-continuous uninorm with neutral element $u < 1$, and let f be the interpretation of $\overline{0}$. Then, an admissible evaluation would be given by:

$$\|Ax\| = \begin{cases} (u-1)x/90 + 1, & \text{if } 0 \leq x < 90 \\ -fx/270 + 4f/3, & \text{if } 90 \leq x < 360 \end{cases}$$

that is, a piecewise linear map that maps all acute angles to the interval of designated elements $[u, 1]$ (in particular it maps 0 to 1, because $0°$ is the most acute angle), and maps all non-acute angles to the interval $[0, f]$ (in particular, it maps $90°$ to f, because it is the least non-acute angle). Observe that no angle is mapped to the interval (f, u) of intermediate truth-values. This will be a common characteristic of all models for graded precise predicates, because, unlike vague predicates, they have no borderline cases. Again, MFL offers a wealth of logical systems to model a multitude of graded precise predicates depending on the needs of each context.

This semantical treatment of graded precise predicates is inspired by Paoli's proposal [26, 27] where the interpretations were given on algebraic models of Casari's comparative logic.

6 A General Program

We propose a research program of correct reasoning with graded properties, done from the point of view of Logic and based on the following three layers of analysis:

1. *Natural language and natural reasoning scenarios:* Interdisciplinary research relating Logic to Cognitive Science, Psychology and Linguistics in order to understand how correct reasoning is actually carried out in natural language with graded properties.
2. *Formal interpreted languages and artificial reasoning scenarios:* The application of tools of mathematical logic (level 3, below) to natural reasoning scenarios (level 1, above) requires the introduction of a middle level, in which the logical formalisms come with specific interpretations of graded properties in specific contexts. This has the potential for applications to Computer Science that require handling graded predicates.
3. *Formal abstract languages and mathematical logic:* The systematic mathematical study of non-classical logical systems with a graded semantics upon which the study of the previous levels can be based.

The ideas sketched in the previous section illustrate how the mathematical machinery developed in the study of non-classical logics can be used in modelling graded predicates and reasoning with them. The study of such logical systems in layer three is done in a completely abstract way, as free mathematical research unconstrained by the possible interpretations of the formal language.

However, by requiring specific behaviour of (part of) the formal language, motivated by particular reasoning scenarios, we move to the second layer: for example, studying logics with additional modalities aimed at modelling agents' partial knowledge or belief, or their degrees of preference—or other specific fragments of first-order or higher-order logics with good representational power and complexity properties. The potential for applications is suggested by the fact that some areas of Computer Science use several kinds of *weighted* notions, i.e. graded properties, for example in valued constraint satisfaction problems, weighted graphs, weighted automata, and so on. The application of MFL and other algebraic logical tools to these areas is still very much under-explored.

Finally, the first layer can be seen as a proposal in the spirit of Stenning and van Lambalgen's endeavour to bring Logic back to the study of reasoning [35], after years of evolving in separate directions. They have convincingly argued that Logic is very much domain-dependent: valid forms of inference depend on the domain of discourse. Accordingly, they claim that each instance of reasoning requires two stages:

1. *reasoning to an interpretation:* in which one has to decide what are the appropriate formal tools for the particular reasoning scenario (language, models, notion of consequence)
2. *reasoning from an interpretation:* in which, having established the previous parameters, one can reason according to the chosen form of inference.

This approach is compatible with the kind of plurivaluationism defended in [33] and with the idea, advocated in [1] and mentioned in Sect. 4 above, that we need not pick one particular logic from the MFL family and then use it in every reasoning context that involves graded notions: there are differences among graded notions (e.g. vague vs graded precise—and within the vague predicates, linear vs nonlinear; etc.) and different contexts may well require different logics. In general, our broader aim is to apply the full suite of MFL tools, the plurivaluationism of [33], and the methodology proposed in [35] to reasoning scenarios involving graded properties.

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References

1. Běhounek, L.: In which sense is fuzzy logic a logic for vagueness? In: Lukasiewicz, T., Peñaloza, R., Turhan, A.Y. (eds.) Proceedings of the First Workshop on Logics for Reasoning about Preferences, Uncertainty, and Vagueness (PRUV 2014), Vienna, pp. 26–39 (2014)
2. Běhounek, L., Cintula, P.: Fuzzy logics as the logics of chains. Fuzzy Sets Syst. **157**(5), 604–610 (2006)
3. Blok, W.J., Pigozzi, D.L.: Algebraizable Logics, Memoirs of the American Mathematical Society, vol. 396. American Mathematical Society, Providence (1989). http://orion.math.iastate.edu/dpigozzi/
4. Casari, E.: Comparative logics and Abelian ℓ-groups. In: Logic Colloquium 1988, Padova. Studies in Logic and the Foundations of Mathematics, vol. 127, pp. 161–190. North-Holland, Amsterdam (1989)
5. Cignoli, R., Esteva, F., Godo, L., Torrens, A.: Basic fuzzy logic is the logic of continuous t-norms and their residua. Soft. Comput. **4**(2), 106–112 (2000)
6. Cintula, P., Fermüller, C.G., Hájek, P., Noguera, C. (eds.): Handbook of Mathematical Fuzzy Logic (in three volumes). Studies in Logic, Mathematical Logic and Foundations, vols. 37, 38, and 58. College Publications (2011, 2015)
7. Cintula, P., Horčík, R., Noguera, C.: Non-associative substructural logics and their semilinear extensions: axiomatization and completeness properties. Rev. Symbolic Logic **6**(3), 394–423 (2013)
8. Cintula, P., Horčík, R., Noguera, C.: The quest for the basic fuzzy logic. In: Montagna, F. (ed.) Petr Hájek on Mathematical Fuzzy Logic. OCL, vol. 6, pp. 245–290. Springer, Cham (2015). doi:10.1007/978-3-319-06233-4_12
9. Cintula, P., Noguera, C.: Implicational (semilinear) logics I: a new hierarchy. Arch. Math. Logic **49**(4), 417–446 (2010)
10. Cintula, P., Noguera, C.: A Henkin-style proof of completeness for first-order algebraizable logics. J. Symb. Log. **80**(1), 341–358 (2015)
11. Dummett, M.: A propositional calculus with denumerable matrix. J. Symb. Log. **24**(2), 97–106 (1959)
12. Esteva, F., Godo, L.: Monoidal t-norm based logic: towards a logic for left-continuous t-norms. Fuzzy Sets Syst. **124**(3), 271–288 (2001)
13. Esteva, F., Godo, L., García-Cerdaña, À.: On the hierarchy of t-norm based residuated fuzzy logics. In: Fitting, M.C., Orlowska, E. (eds.) Beyond Two: Theory and Applications of Multiple-Valued Logic, Studies in Fuzziness and Soft Computing, vol. 114, pp. 251–272. Springer, Heidelberg (2003)
14. Fine, K.: Vagueness, truth and logic. Synthese **30**, 265–300 (1975). Reprinted with corrections in [22], pp. 119–150
15. Galatos, N., Jipsen, P., Kowalski, T., Ono, H.: Residuated Lattices: An Algebraic Glimpse at Substructural Logics. Studies in Logic and the Foundations of Mathematics, vol. 151. Elsevier, Amsterdam (2007)
16. Galatos, N., Ono, H.: Cut elimination and strong separation for substructural logics: an algebraic approach. Ann. Pure Appl. Logic **161**(9), 1097–1133 (2010)
17. Hájek, P.: Metamathematics of Fuzzy Logic, Trends in Logic, vol. 4. Kluwer, Dordrecht (1998)
18. Hájek, P.: Fuzzy logics with noncommutative conjunctions. J. Log. Comput. **13**(4), 469–479 (2003)
19. Jenei, S., Montagna, F.: A proof of standard completeness for Esteva and Godo's logic MTL. Stud. Logica. **70**(2), 183–192 (2002)

20. Jenei, S., Montagna, F.: A proof of standard completeness for non-commutative monoidal t-norm logic. Neural Netw. World **13**(5), 481–489 (2003)
21. Keefe, R.: Theories of Vagueness. Cambridge University Press, Cambridge (2000)
22. Keefe, R., Smith, P. (eds.): Vagueness: A Reader. MIT Press, Cambridge (1997)
23. Lambek, J.: On the calculus of syntactic types. In: Jakobson, R. (ed.) Structure of Language and Its Mathematical Aspects, pp. 166–178. American Mathematical Society, Providence (1961)
24. Łukasiewicz, J., Tarski, A.: Untersuchungen über den Aussagenkalkül. Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, cl. III **23**(iii), 30–50 (1930)
25. Metcalfe, G., Montagna, F.: Substructural fuzzy logics. J. Symb. Log. **72**(3), 834–864 (2007)
26. Paoli, F.: Truth degrees, closeness, and the sorites. In: Bueno, O., Abasnezhad, A. (eds.) On the Sorites Paradox. Springer (to appear)
27. Paoli, F.: Comparative logic as an approach to comparison in natural language. J. Semant. **16**, 67–96 (1999)
28. Raffman, D.: Vagueness without paradox. Philos. Rev. **103**, 41–74 (1994)
29. Shapiro, S.: Vagueness in Context. Oxford University Press, Oxford (2006)
30. Smith, N.J.J.: Consonance and dissonance in solutions to the sorites. In: Bueno, O., Abasnezhad, A. (eds.) On the Sorites Paradox. Springer (to appear)
31. Smith, N.J.J.: Undead argument: the truth-functionality objection to fuzzy theories of vagueness. To appear in Synthese. doi:10.1007/s11229-014-0651-7
32. Smith, N.J.J.: Vagueness as closeness. Australas. J. Philos. **83**, 157–183 (2005)
33. Smith, N.J.J.: Vagueness and Degrees of Truth. Oxford University Press, Oxford (2008). Paperback 2013
34. Sorensen, R.: Vagueness and Contradiction. Oxford University Press, Oxford (2001)
35. Stenning, K., van Lambalgen, M.: Human Reasoning and Cognitive Science. MIT Press, Cambridge (2008)
36. Williamson, T.: Vagueness. Routledge, London (1994)