Vagueness by Numbers? No Worries

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Theories of vagueness based upon fuzzy set theory countenance a continuum of *degrees of truth*, usually represented by the real numbers between 0 and 1 inclusive (with 0 representing falsity *simpliciter* and 1 representing truth *simpliciter*). Keefe (1998) argues against such theories of vagueness.[1] She writes:

> Take the vague predicate 'tall': I claim that any numbers assigned in an attempt to capture the vagueness of 'tall' do no more than serve as another measure of *height*. More generally, in so far as it is possible to assign numbers which respect certain truths about, for example, comparative relations, this is no more than a measure of an attribute related to, or underlying, the vague predicate. (Keefe 1998, p. 575)

The purpose of this note is to show that Keefe's claim here is false: the numbers that we assign to objects to measure their heights serve a quite distinct purpose from the numbers that the fuzzy theory assigns to objects to measure their degrees of tallness; and in general, the numbers that we assign to objects to measure their possession of a quantity Q serve a quite distinct purpose from the numbers that the fuzzy theory assigns to objects to measure their degree of possession of property P, where an object's possession of P is determined by its possession of Q (so Q might be mass and P the property of being heavy, Q might be hair-count and P the property of being bald, and so on).[2]

Keefe notes that 'In many paradigm cases of a vague predicate F there is a corresponding measurable attribute related to F in such a way that the truth-value status of Fx … is determined by x's quantity of that attribute. For example, the truth-value status of 'a is tall' is determined by, or supervenes on, a's height … similarly for the relation between 'a

[1] Keefe (1998) appears in a slightly revised form as Ch. 5 of Keefe (2000).

[2] Of course, many properties P are 'multi-dimensional': whether or not (or to what degree) an object possesses P is determined by the object's possession of a *number* of quantities $Q_1,\ldots,Q_n$. This does not affect my point: the discussion here applies, *mutatis mutandis*, to multi-dimensional predicates.

Mind, Vol. 112 . 446 . April 2003

is hot' and *a*'s temperature' (Keefe 1998, p. 575). I agree. Keefe continues:

> But although the measure of the underlying quantity may *determine* the applicability of the vague predicate, it does not follow that this measure is reflected in non-classical numerical truth-values … Are degree theorists thus mistaken in claiming that vague predicates come in degrees? I suggest that there is a *sense* in which *F* can be said to come in degrees—call it coming in degrees_m—whenever there is a measure of the attribute *F*-ness, and where things have different degrees_m of *F*-ness by having more or less of the attribute. The degree_m of heat of an object will be a matter of its quantity of heat and we happen to call the measure *degrees* Celsius … But the fact that many vague predicates come in degrees_m is not enough for the degree theorist, who needs there to be implications for truth-values or degrees of truth, so that if *F* comes in degrees, predications of *F* can be true to intermediate degrees … coming in degrees_m is not the sense of 'coming in degrees' required by the degree theorist. (Keefe 1998, pp. 575–6)

Again, I agree. So far we have no objection to the fuzzy view: we have a warning not to confuse degrees of truth with degrees_m, but we have no argument to the effect that the fuzzy theory is inextricably entangled in such confusion. Thus, it is surprising that Keefe continues, 'The confusion between the different senses of "coming in degrees" can be further illustrated by reference to a common argument aiming to show that we must adopt a degree theory of vagueness' (Keefe 1998, p. 576). Suddenly Keefe has gone from drawing a correct distinction—between degrees_m, and degrees of truth as the fuzzy theory conceives of them—to claiming that the fuzzy theory ignores this distinction. This is too swift: certain fuzzy theorists may have been involved in such a confusion, but this does not mean that the fuzzy position itself is essentially confused.

Some fuzzy theorists *do* confuse the very things Keefe warns us to keep apart. Keefe (1998, pp. 576–7) attributes the following argument to Forbes (1983, pp. 241–2):

> Consider a pair of people, *a* and *b*, such that
> (1) *a* is taller than *b*. We can infer
> (2) *a* is tall to a greater degree than *b*; so
> (3) *a* satisfies the predicate 'is tall' to a greater degree than *b*; and hence
> (4) '*a* is tall' has a higher degree of truth than '*b* is tall'.

This argument is a bad one, as Keefe points out: with the 'degrees_m' sense of 'degree' in play, (2) follows from (1), but (3) and (4) do not follow from (2); whereas with the 'degrees of truth' sense of 'degrees' in play, (4) follows from (2), but (2) does not follow from (1). But from the fact that some fuzzy *theorists* are guilty of confusion, it does not follow that the fuzzy *theory* is essentially confused. Keefe has given us no reason to conclude the latter—and furthermore, the conclusion is false, as I shall now show.

We have two words: 'tall' and 'taller'.[3] There is certainly some important connection between these two words, but it is not totally straightforward. It is certainly *not* the case that if a is taller than b, then 'a is tall' is truer than 'b is tall'. This claim is not part of the fuzzy view (at least, it *should* not be—but as we have seen, some fuzzy theorists are indeed confused on this point). Nevertheless, there *is* room for the idea that sentences of the form 'a is tall' might be true to intermediate degrees: just because the simple-minded route to this idea is mistaken (that is, the route rejected in the previous paragraph), it does not mean that there is no route to this idea.

The clearest way to think of matters in this area is as follows. First, there are objects that have heights: persons, mountains, and so on. Then, there are the heights that these things have: these heights are also objects. So we have two sets: a set O of persons, mountains, and so on; and a set H of heights, which is equipped with an ordering relation. There is a mapping h from O to H, which assigns to each object its height. There is also a third set of objects: the set $\mathbf{R}$ of real numbers. There are various mappings from the set of heights to the set of real numbers; each of these may be thought of as giving a name to each height. Suppose that Bob's height is x, that is, $h(\text{Bob}) = x$. One mapping f from the set of heights to the set of reals assigns x the number 6; intuitively, $f(h(\text{Bob}))$ is Bob's height in *feet*. Another mapping m from the set of heights to the set of reals assigns x the number 1.8; intuitively, $m(h(\text{Bob}))$ is Bob's height in *metres*. A third mapping c from the set of heights to the set of reals assigns x the number 180; intuitively, $c(h(\text{Bob}))$ is Bob's height in *centimetres*; and so on. There are familiar relations between these mappings; for example, $c(x)=30f(x)$ ('There are thirty centimetres in a foot').

The situation with regard to 'taller' is straightforward. For any objects x and y in O, x is taller than y just in case $h(y)<h(x)$ (that is, $h(y) \leq h(x)$ and not $h(x) \leq h(y)$). But what about 'tall'? As a first try, we might say that there is a distinguished subset T of H, such that for any object x in O, x is tall just in case $h(x) \in T$. The idea is that x is tall just in case x is of a sufficient height. Implicit in the word 'sufficient' here is the idea that T should be closed upwards: for any x and y in H, if $x \leq y$ and $x \in T$, then $y \in T$. This immediately gives us an important relation

[3] This is intended as an illustrative example: the following discussion applies, *mutatis mutandis*, to other such pairs of words also, e.g. 'loud' and 'louder', 'heavy' and 'heavier', etc.

between 'tall' and 'taller': for any x and y in O, if x is taller than y and y is tall, then x is tall.[4]

So far so good—and no degrees of truth in sight (only degrees$_m$). But there is something wrong with this model: it ignores the vagueness of 'tall'.[5] Intuitively, if two objects a and b in O are very close in respect of height, then 'a is tall' and 'b is tall' are very close in respect of truth. In the picture outlined above, however, assuming that O contains a series of objects ranging from one that is not tall to one that is tall, in very small steps of height, there will be a pair of things a and b in O whose heights are very close, one of which is tall and the other is not—that is, 'a is tall' is true *simpliciter* and 'b is tall' is false *simpliciter*. Thus the proposed picture does not allow for the vagueness of 'tall'. In response to this problem, the fuzzy theory proposes that we replace the classical subset T of H with a *fuzzy* subset T, and modify the requirement that T be upward closed to the requirement that for any x and y in H, if $x \leq y$ then x's degree of membership in T is less than or equal to y's degree of membership in T. Now, 'a is tall' will be true to whatever degree $h(a)$ is in T, and thus we have the following important relation between 'tall' and 'taller': for any x and y in O, if x is taller than y, then the degree of truth of 'x is tall' is at least as great as the degree of truth of 'y is tall'. We now have the resources to accommodate the vagueness of 'tall' (if a and b in O are very close in respect of height, then it *can* now be the case that 'a is tall' and 'b is tall' are very close in respect of truth), and we are *not* committed to the idea that if a is taller than b, then 'a is tall' is truer than 'b is tall'—that is, we can also accommodate the intuitive idea that while Kareem Abdul Jabbar is taller than Larry Bird, 'Kar-

[4] An anonymous referee pointed out here that nothing is tall *simpliciter*, but rather tall *for an F*. In order to accommodate this observation, we would need—instead of a single distinguished subset T of H—different subsets T_F for different kinds F of thing, with x being tall for an F just in case $h(x) \in T_F$. In order to avoid complexities of formulation that do not bear on my central point, I leave it to the reader to add this sort of qualification throughout: where I write simply of tallness or the subset T, read this as talk of tallness for an arbitrarily chosen kind F of thing.

[5] At any rate, this is what the fuzzy theorists think. Whether or not they are actually *correct* here is irrelevant to the issue under discussion: the issue of whether the fuzzy view is essentially *confused*. What follows is a brief presentation of the motivation for the fuzzy view; these issues are discussed in detail in Smith (2001). In particular, I argue there for the following *definition* of vagueness. A predicate 'P' is vague if and only if it satisfies the following condition (for any objects a and b):

Closeness If a and b are very similar in P-relevant respects, then 'Pa' and 'Pb' are very similar in respect of truth.

(In terms of the present discussion, two objects are very similar in P-relevant respects if they possess very similar quantities of Q, where Q is the measurable quantity underlying possession of the property picked out by 'P'.) I then argue that one of the big advantages of the fuzzy view of vagueness is that it can accommodate vagueness as characterized in terms of Closeness.

eem Abdul Jabbar is tall' is *not* truer than 'Larry Bird is tall', for both sentences are true *simpliciter*.[6]

In the first picture (where T is a classical subset of H), we have degrees$_m$ of height and no degrees of truth. In the second picture (where T is a fuzzy subset of H), we have degrees$_m$ of height *and* degrees of truth of sentences of the form '*a* is tall'. Thus, in the second picture we have what Keefe says we cannot have: numbers assigned in an attempt to capture the vagueness of 'tall' which do *not* simply serve as another measure of height. In the second picture, we have maps f from H to **R**, and then composite maps $f \circ h$ from O to **R** which serve as measures of height.[7] We also have something entirely distinct: a fuzzy subset T of H, or (identifying T with its characteristic function) a map T from H to $[0,1]$, and then a composite map $T \circ h$ from O to $[0,1]$, which captures the vagueness of 'tall', and respects the comparative relation that if x is taller than y, then the degree of truth of 'x is tall' is at least as great as the degree of truth of 'y is tall'. These maps are formally and conceptually distinct, and there is no reason why we cannot have both.[8]

At this point, I have (I believe) done what I set out to do: shown that the fuzzy view of vagueness does not involve a confusion between degrees$_m$ and degrees of truth. There are, however, two further points which are worth discussing.

First, something which probably contributes to the view that the fuzzy theory confuses degrees$_m$ and degrees of truth is the fact that the two maps which I have distinguished ($f \circ h$ and $T \circ h$) both assign *real numbers* to objects in O (in the case of $T \circ h$, real numbers confined to the interval $[0,1]$). This fact should not mislead us into ignoring the differences between $f \circ h$ and $T \circ h$; but if one does find this fact (potentially) confusing, then one should note that the fuzzy theory should *not* in fact *identify* its degrees of truth with the reals in $[0,1]$. When we talk of the 'real numbers', we invoke *two* things: the *order-type* of the reals (lacking a first or last element, being dense, being complete, and having a countable order-dense subset); and the *algebra* of the reals (together with the usual operations of addition and multiplication, the set of real numbers

[6] Kareem Abdul Jabbar is 7'2" in height and Larry Bird is 6'9" in height. Thanks to Scott Soames for this example.

[7] The symbol '$\circ$' denotes composition of functions; $(f \circ h)(x) = f(h(x))$, i.e. it is what you get if you do h to x, and then do f to the result.

[8] An anonymous referee asked 'Why aren't the values of $T \circ h$ another measure of height?' Well, because it might (indeed should) be the case that $T \circ h$(Kareem Abdul Jabbar) = $T \circ h$(Larry Bird) (= 1), but it is *not* the case that Kareem Abdul Jabbar and Larry Bird have the same height.

forms a *field*). The fuzzy theory, however, simply wants a set of truth values with the *order-type* of [0,1]; it does *not* also want operations of addition and multiplication on this set satisfying the field axioms (it wants *different* operations, corresponding to the logical connectives).[9] Thus, when we talk of [0,1] in connection with the fuzzy theory, we should be thinking not of the real interval [0,1] itself, but of a set of truth values which simply has the same order-type as [0,1]. Once we make this distinction, it should be even harder to miss the differences between *foh* and *Toh*.[10]

Second, in the theory of measurement, the set *H* of heights is often ignored: we deal directly with mappings from the set *O* of objects that have heights to the set **R** of real numbers.[11] In some contexts this makes things simpler, but in the present context my aim has been to be very clear about the relationship between 'tall' and 'taller', and it seems to me that we only muddy the waters if we leave *heights* out of the picture. We certainly appear to believe in such things: we often refer to them, for example when we say that Bill's height is greater than Bob's. Those with nominalist leanings will, of course, want to explain away such talk—but as is often the case in these disputes, we get a clearer picture if we take our talk at face value.[12] Nevertheless, my point can be made *without* countenancing heights as objects, and I shall conclude by indicating how this can be done. If we ignore the set *H* of heights, then we have maps from *O* to **R**, which assign heights to objects (these heights now being thought of simply as real numbers). There are various maps, one giving the heights of objects in metres, one giving the heights of objects in feet, and so on. For the sake of convenience, let us fix on one such map *h*. In Keefe's terminology, *h* assigns degrees$_m$ of tallness to objects. The situation with regard to 'taller' is now straight-

[9] One exception here is Goguen (1968–9), who recommends that we use *multiplication* to model *conjunction*.

[10] Note that my point here is quite different from that of those (e.g. Goguen 1967) who criticize the fuzzy theory's choice of [0,1] as its set of truth values, on the grounds that the truth values should be merely partially—not linearly—ordered.

[11] For details on measurement theory see Suppes and Zinnes (1963) and Coombs *et al.* (1954), or the work by Krantz *et al.* cited by Keefe.

[12] I would like to thank Amitavo Islam for first introducing me, many years ago, to this conception of heights, and other quantities, as objects. This conception is often met with blank incomprehension by philosophers, whilst amongst many mathematicians the conception is considered too obviously correct even to be worth stating explicitly, let alone debating.

forward. For any objects x and y in O, x is taller than y just in case $h(y)<h(x)$.[13] Turning to 'tall', as a first try we might say that there is a distinguished subset T of $\mathbf{R}$, such that for any object x in O, x is tall just in case $h(x)\in T$. T will be closed upwards, so that for any x and y in O, if x is taller than y and y is tall, then x is tall. For reasons discussed above, however, the fuzzy theorist thinks that this picture ignores the vagueness of 'tall', and so proposes replacing the classical subset T of $\mathbf{R}$ with a *fuzzy* subset T, that is, a map T from $\mathbf{R}$ to [0,1]. The requirement that T be upwards closed becomes the requirement that for any x and y in $\mathbf{R}$, if $x \leq y$ then $T(x) \leq T(y)$. In this picture we have two maps, $h: O \rightarrow \mathbf{R}$ which assigns degrees$_m$ of tallness to objects, and $T \circ h: O \rightarrow [0,1]$ which assigns degrees of tallness (in the degrees of truth sense) to objects. As before, these maps are formally and conceptually distinct, and there is no reason why we cannot have both.

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[13] It does not matter here that we fixed on one particular map h, because height is measured on a *ratio scale*, meaning that for *any* height-measuring map $j: O \rightarrow \mathbf{R}$, there is a positive real number α such that for any object x in O, $j(x)=\alpha h(x)$; thus if $h(y)<h(x)$, then it is also the case that $j(y)<j(x)$.

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